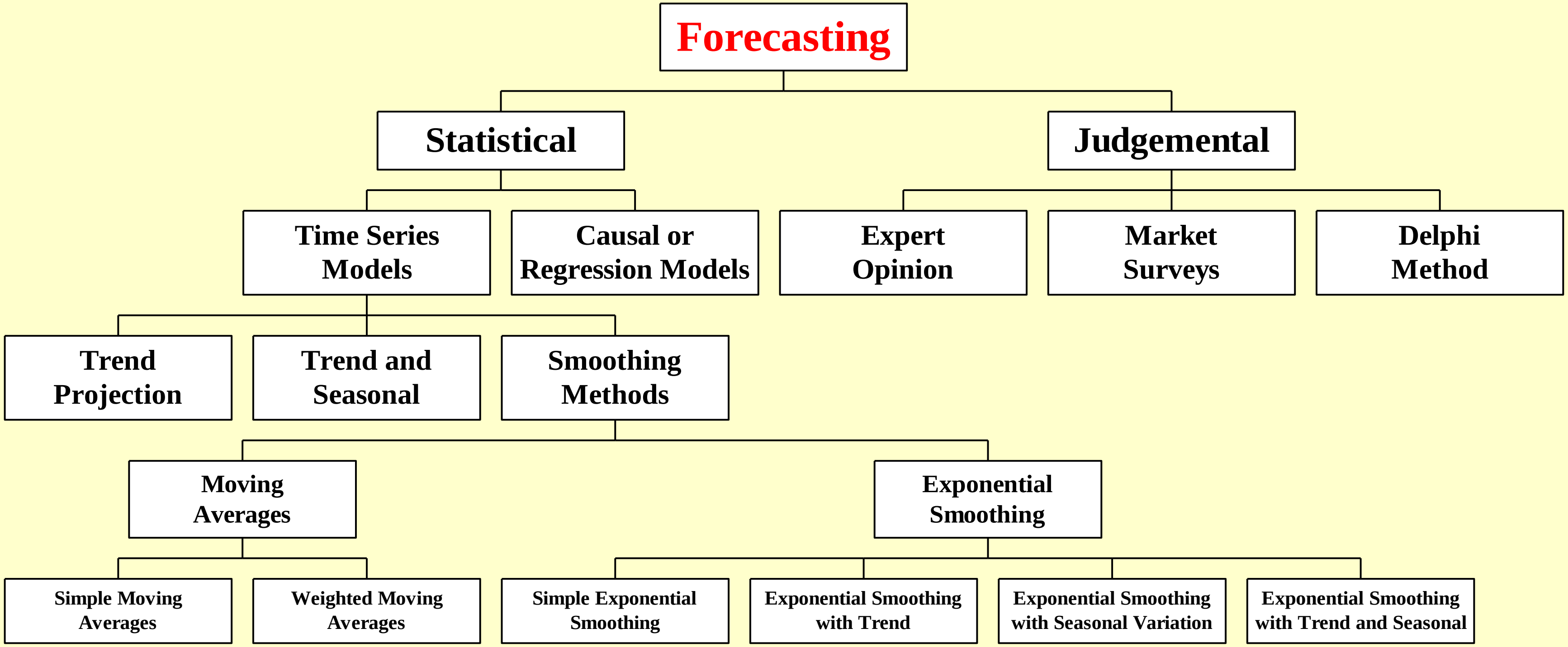
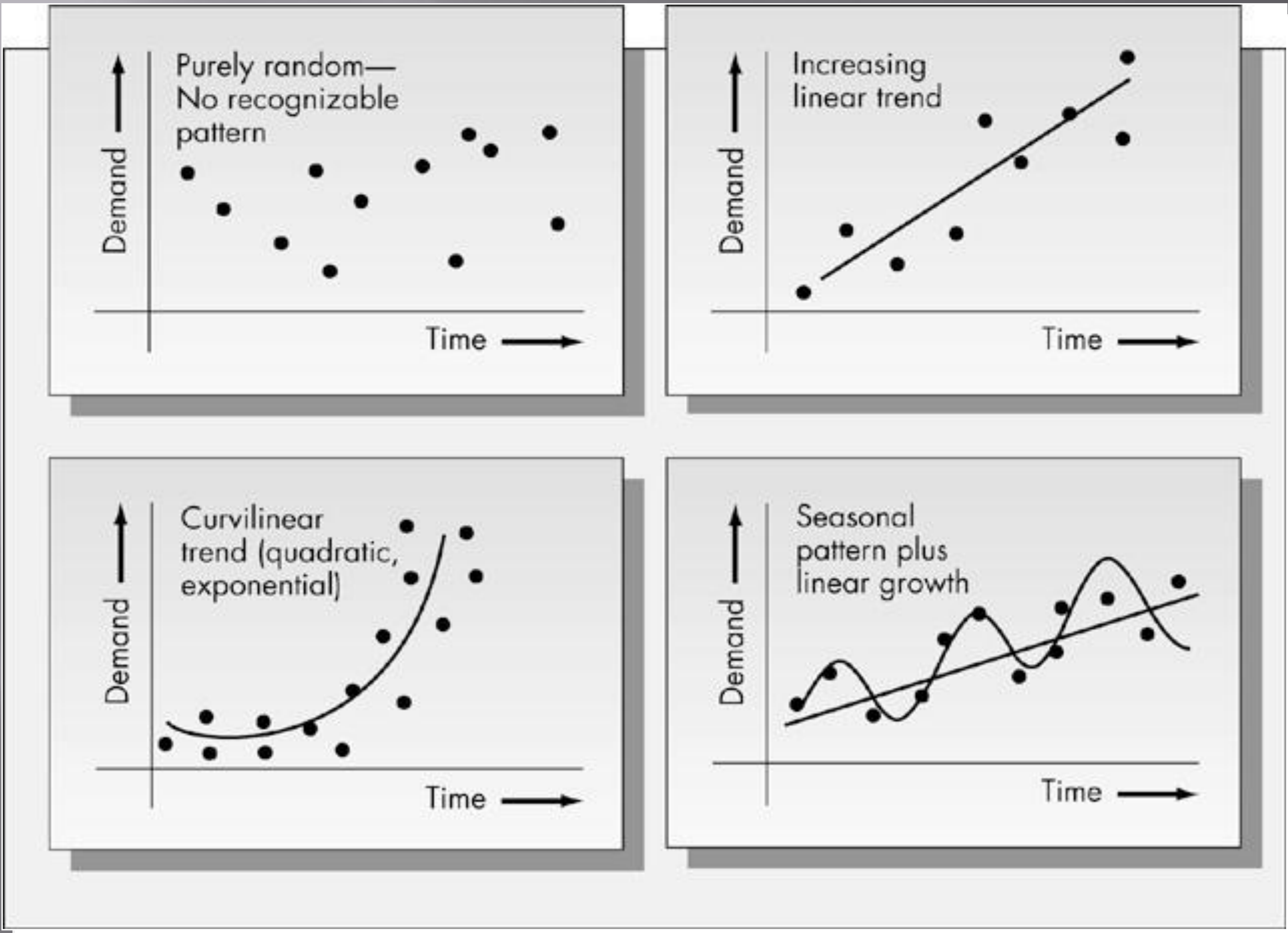
PRODUCTION PLANNING AND CONTROL

Lecture 2

Classification of Forecasting Models




Purely random—
No recognizable
pattern
Demand
Time
Increasing
linear trend
Demand
Time
Curvilinear
trend (quadratic,
exponential)
Demand
Time
Seasonal
pattern plus
linear growth
Demand
Time

Difference between interpolation and extrapolation???

Interpolation

For example, suppose we wanted the square root of 2.155 and had a table such as shown below:

n	$\sqrt{n}$
2.140	1.46287
2.150	1.46629
2.160	1.46969
2.170	1.47309

The difference between the square roots of 2.15 and 2.16 is 0.0034. Since 2.155 is half-way between 2.15 and 2.16 we could assume that its square root was half-way between the square roots of 2.15 and 2.16. This gives 2.155 = 1.46799.

Had we wanted 2.153 we would have added 0.3 times 0.0034 to 2.15. This is the process of ***LINEAR INTERPOLATION***, in which we assume a linear variation between the two known values to predict intermediate values.

Newton's Forward Difference Interpolation Formula

$$y_{r+1} - y_r$$

is called the first difference of y_r, denoted by Δy_r: $\Delta y_r = y_{r+1} - y_r$

Similarly, the relationship $\Delta y_{r+1} - \Delta y_r$ is the ***second difference:*** $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$

For example, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

Similarly $\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$

In general terms: $\Delta^2 y_r = y_{r+2} - 2y_{r+1} + y_r \qquad r = 0, 1, 2, \ldots, n-2$

The ***third differences*** are $\Delta^3 y_r = y_{r+3} - 3y_{r+2} + 3y_{r+1} - y_r \qquad r = 0, 1, 2, \ldots, n-3$

Proof of forward difference formula

Statement: If $x_0, x_1, x_2, \ldots, x_n$ are given set of observations with common difference h and let $y_0, y_1, y_2, \ldots, y_n$ are their corresponding values, where $y = f(x)$ be the given function then

$$f(x) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\,\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ldots + \frac{p\,(p-1)(p-2)\ldots(p-(n-1))}{n!}\,\Delta^n y_0$$

Proof: Let us assume an n^{th} degree polynomial

$$f(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1) + \ldots + A_n\,(x - x_0)(x - x_1)\ \ldots (x - x_{n-1}) \dashrightarrow \text{(i)}$$

Newton forward difference interpolation derivation

First Step: Find coefficients A0, A1, A2..An

Substitute $x = x_0$ in (i), we get $f(x_0) = A_0 \implies y_0 = A_0$

Substitute $x = x_1$ in (i), we get $f(x_1) = A_0 + A_1(x_1 - x_0) \implies y_1 = y_0 + A_1 h$

$$\implies A_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

Substitute $x = x_2$ in (i), we get $f(x_2) = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$

$$\implies y_2 = y_0 + A_1(2h) + A_2(2h)(h)$$

$$\implies y_2 = y_0 + 2h\left(\frac{\Delta y_0}{h}\right) + 2h^2 A_2$$

$$\implies A_2 = \frac{1}{2h^2}\Delta^2 y_0$$

Similarly, we get $A_n = \frac{1}{nh^2}\Delta^n y_0$

Newton forward difference interpolation derivation

Substituting these values in (i), we get

$$f(x) = y_0 + (x - x_0)\frac{1}{h}\Delta y_0 + (x - x_0)(x - x_1)\frac{1}{2h^2}\Delta^2 y_0 + \ \dots + (x - x_0)(x - x_1)\ \dots (x - x_{n-1})\frac{1}{nh^2}\Delta^n y_0$$

----(ii)

But given $p = \frac{x-x_0}{h}$

$\Longrightarrow x - x_0 = ph \Longrightarrow x = x_0 + h$

$\Longrightarrow\ x - x_1 = x - (x_0 + h)$

$= (x - x_0) - h$

$= ph - h = (p-1)h$

Similarly, $x - x_2 = (p-2)h$,

$\vdots$

$x - x_{n-1} = (p - (n-1))h$

Newton forward difference interpolation derivation

Substituting in the Equation (ii), we get

$$f(x) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\,\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ \dots + \frac{p\,(p-1)(p-2)\dots(p-(n-1))}{n!}\,\Delta^n y_0$$

Exercise

Find a cubic polynomial in x which takes on the values -3, 3, 11, 27, 57 and 107, when $x = 0, 1, 2, 3, 4$ and 5 respectively.

Solution

Here, the observations are given at equal intervals of unit width.

To determine the required polynomial, we first construct the difference table

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3	6		
1	3	8	2	
2	11	16	8	6
3	27	30	14	6
4	57	30	20	6
5	107	50		

Since the 4th and higher order differences are zero, the required Newton's interpolation formula

$$f(x_0+ph)=f(x_0)+p\Delta f(x_0)+\frac{p(p-1)}{2}\Delta^2 f(x_0)$$

$$+\frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0)$$

Here,

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

$$\Delta f(x_0) = 6$$

$$\Delta^2 f(x_0) = 2$$

$$\Delta^3 f(x_0) = 6$$

Substituting these values into the formula, we have

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

$$f(x) = x^3 - 2x^2 + 7x - 3,$$

The required cubic polynomial.

NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

Newton backward difference interpolation derivation

Statement: If $x_0, x_1, x_2, \dots, x_n$ are given set of observations with common difference h and let $y_0, y_1, y_2, \dots, y_n$ are their corresponding values, where $y = f(x)$ be the given function then

$$f(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\,\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\,\nabla^3 y_0 + \,\dots + \frac{p\,(p+1)(p+2)\dots(p+(n-1))}{n!}\,\nabla^n y_0$$

where $p = \frac{x - x_0}{h}$

Proof: Let us assume an n^{th} degree polynomial

$$f(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + \,\dots + A_n\,(x - x_n)(x - x_{n-1})\,\dots(x - x_1)$$

--> (i)

Newton backward difference interpolation derivation

Substitute $x = x_n$ in (i), we get $f(x_n) = A_0 \implies y_n = A_0$

Substitute $x = x_{n-1}$ in (i), we get $f(x_{n-1}) = A_0 + A_1(x_{n-1} - x_n) \implies y_{n-1} = y_n - A_1 h$

$$\implies A_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_0}{h}$$

Substitute $x = x_{n-2}$ in (i), we get $f(x_{n-2}) = A_0 + A_1(x_{n-2} - x_n) + A_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$

$$\implies y_{n-2} = y_n + A_1(-2h) + A_2(-2h)(-h)$$

$$\implies y_{n-2} = y_n - 2h\left(\frac{\nabla y_n}{h}\right) + 2h^2 A_2$$

$$\implies A_2 = \frac{1}{2h^2}\nabla^2 y_n$$

Similarly, we get $A_n = \frac{1}{nh^2}\nabla^n y_n$

Newton backward difference interpolation derivation

Substituting these values in (i), we get

$$f(x) = y_n + (x - x_n)\frac{1}{h}\nabla y_n + (x - x_n)(x - x_{n-1})\frac{1}{2h^2}\nabla^2 y_n + \ \dots$$

$$+(x - x_n)(x - x_{n-1}) \ \dots (x - x_1)\frac{1}{nh^2}\nabla^n y_n \qquad \text{---- (ii)}$$

But given $p = \frac{x - x_n}{h}$

$\Longrightarrow x - x_n = ph \Longrightarrow x = x_n + h$

$\Longrightarrow \ x - x_{n-1} = x - (x_n - h)$

$= (x - x_n) + h$

$= ph + h = (p + 1)h$

Similarly, $x - x_{n-2} = (p + 2)h$,

$\vdots$

$x - x_1 = (p + (n - 1))h$

Newton backward difference interpolation derivation

Substituting in the Equation (ii), we get

$$f(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\,\Delta^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\,\Delta^3 y_n + \ \dots + \frac{p\,(p+1)(p+2)\dots(p+(n-1))}{n!}\,\Delta^n y_n$$

This formula is known as Newton’s backward interpolation formula. This formula is also known as Newton’s-Gregory backward difference interpolation formula.

Example

For the following table of values, estimate *f* (7.5).

x	1	2	3	4	5	6	7	8
$y = f(x)$	1	8	27	64	125	216	343	512

Solution

The value to be interpolated is at the end of the table. Hence, it is appropriate to use Newton's backward interpolation formula. Let us first construct the backward difference table for the given data

Difference Table

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
2	8	7			
3	27	19	12		
4	64	37	18	6	
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0

Since the 4th and higher order differences are zero, the required Newton's backward interpolation formula is

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

In this problem,

$$p = \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5$$

$$\nabla y_n = 169, \quad \nabla^2 y_n = 42, \quad \nabla^3 y_n = 6$$

$$y_{7.5} = 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2}(42)$$

$$+ \frac{(-0.5)(0.5)(1.5)}{6}(6)$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.875$$

Example

The sales for the last five years is given in the table below. Estimate the sales for the year 1979

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Solution
Newton’s backward difference table for the given data as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
1976	43	3			
1978	48	5	2		
1980	52	4	-1	-3	
1982	57	5	1	2	5

In this example,

$$p = \frac{1979 - 1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1,$$

$$\nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's interpolation formula gives

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1)$$
$$+ \frac{(-1.5)(-0.5)(0.5)}{6}(2)$$
$$+ \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

Therefore, $$y_{1979} = 50.1172$$

Example

Consider the following table of values

x	1	1.1	1.2	1.3	1.4	1.5
F(x)	2	2.1	2.3	2.7	3.5	4.5

Use Newton's Backward Difference Formula to estimate the value of f(1.45) .

x	y=F(x)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	2					
1.1	2.1	0.1				
1.2	2.3	0.2	0.1			
1.3	2.7	0.4	0.2	0.1		
1.4	3.5	0.8	0.4	0.2	0.1	
1.5	4.5	1	0.2	-0.2	-0.4	-0.5

$$p = \frac{x - x_n}{h} = \frac{1.45 - 1.5}{0.1} = -0.5 ,\ \nabla y_n = 1 ,\ \nabla^2 y_n = .2 ,\ \nabla^3 y_n = -.2 ,\ \nabla^4 y_n = -.4 ,$$

$$\nabla^5 y_n = -.5$$

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$
$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$
$$y_x = 4.5 + (-0.5)(1) + \frac{(-0.5)(-0.5+1)}{2!}(0.2) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(-0.2)$$
$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(-0.4) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{5!}(-0.5)$$

$$= 4.5 - 0.5 - 0.025 + 0.0125 + 0.015625 + 0.068359$$
$$= 4.07148$$